QFT I - Problem Set 11

(22) Dirac Bilinears

To caution you from the onset: convince yourself that depending on the sign conventions of the metric (we chose $\eta_{\mu\nu} = diag(-1,1,1,1)$), the Clifford algebra for γ^μ

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

can either yield $(\gamma^0)^2 = -1$ (as it does for us) or $(\gamma^0)^2 = 1$. Correspondingly, in our conventions $(\gamma^0)^\dagger = -\gamma^0$, i.e. γ^0 is anti-hermitian, γ^i hermitian and $(\gamma^i)^2 = 1$. Take a moment to think about the property of γ^i if we had chosen a different sign convention for η. Consulting a book without being aware of the conventions can thus be misleading, because relations might change by a sign.

The Dirac adjoint for a spinor Ψ and a (4x4) matrix A are defined as

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \tag{1}$$
$$\bar{A} = \gamma^0 A^\dagger \gamma^0. \tag{2}$$

a) Show that

$$\overline{AB} = -\bar{B}\bar{A} \tag{3}$$
$$(\bar{\chi} A \Psi)^* = \bar{\Psi}\bar{A}\chi, \tag{4}$$

where χ and Ψ are spinors and A and B arbitrary (4x4) matrices.

b) Show that the condition for the bilinear $\bar{\Psi}A\Psi$ to be real, is

$$\bar{A} = A.$$

Compute that the following matrices fulfill this requirement:

$$\overline{i\mathbf{1}} = i\mathbf{1} \tag{5}$$
$$\bar{\gamma}^5 = \gamma^5 \tag{6}$$
$$\bar{\gamma}^\mu = \gamma^\mu \tag{7}$$
$$\overline{\gamma^\mu\gamma^5} = \gamma^\mu\gamma^5 \tag{8}$$
$$\overline{i\sigma^{\mu\nu}} = i\sigma^{\mu\nu} \tag{9}$$

As in total, there are 16 matrices ($i\mathbf{1}$, 1; γ^5, 1; γ^μ, 4; $\gamma^\mu\gamma^5$, 4; $\sigma^{\mu\nu}$,6), they form a basis for all 4x4 matrices that obey $\bar{A} = A$, i.e. yield real bilinears.

(23) Majorana Fermions

Consider the classical Majorana action

$$S = \int d^4x [i\chi^\dagger \bar{\tau}^\mu \partial_\mu \chi + \frac{im}{2}(\chi^T \tau_2 \chi - \chi^\dagger \tau_2 \chi^*)] \tag{10}$$

where χ are Grassmann valued 2-spinors and $\chi^\dagger := (\chi^*)^T$ and $\bar{\tau}_\mu := (1, -\vec{\tau})$.

a) Show that S is real (*Hint: Remember the complex conjugation for Grassmann numbers a, b: $(ab)^* = b^*a^* = -a^*b^*$.*)

b) Derive the equations of motion for χ and χ^*.

c) Show that the resulting equations of motion are Lorentz invariant, and that they imply the Klein-Gordon equation $(\partial^2 + m^2)\chi = 0$.

d) Now consider the Dirac action

$$S = -i \int d^4x [\bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi] \tag{11}$$

with 4- (Dirac-) spinors $\psi = (\psi_L, \psi_R), \bar{\psi} = \psi^\dagger \gamma^0$. Write $\psi_L = \chi_1, \psi_R = i\tau_2\chi_2^*$ (reflecting that the transformation laws for left- and right-handed spinors are connected by complex conjugation) and express the action in terms of χ_1, χ_2. Compare the resulting Dirac mass term to the Majorana mass term.